# Vegee Brain Automata: Ultradiscretization of essential chaos transversal in neural and ecosystem dynamics

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Abstract. Coupled logistic equations and their discretizations are important models in ecology and complex systems science. However, the chaotic dynamics produced by these nonlinear dynamical systems are lumped together, and the mathematical correspondence between continuous and discrete-time systems is not sufficiently clear. The method of ultradiscretization, which has recently been developed in the analysis of nonlinear integrable systems, can discretize both independent variables such as time and dependent variables such as time-evolving quantities in the dynamical system, while providing an analytical basis for the mathematical correspondence with the original continuous system. In this paper, we first show that the ultradiscretization of the logistic equation has the same form as that of a sigmoidal map, which cannot be derived from a customarily used logistic map. Consequently, recursively coupled systems of sigmoidal functions, such as those employed in neural networks, emerge as new candidate models for various dynamics important in agroecology, where both autonomous dynamics of ecosystems and human intervention could be represented. We then explore qualitative correspondences between neural networks and various modes of farming, including chaotic behavior, and propose an ultra-discretized model that serves as the essential underlying element. The newly proposed model has mathematical connectivity with logistic and tent maps, as well as Holling's disc equations, providing interpretations rooted in ecology and neuroscience. The comprehensive results provide a new perspective for extracting the essence of complex agroecology via computation, which has the potential to link the properties of deep learning being studied in neural networks to the complexity of ecological management.

**Keywords:** Logistic Map, Chaotic Neural Network, Globally Coupled Map Lattices, Ultradiscretizaion, Holling's Disc Equations, Chaotic Itinerancy.

### 1 Introduction

The logistic equation and its coupled Lotka-Volterra equations are a major source of modeling in ecological studies (e.g., [1]). Coupled logistic maps in the form of discrete-time difference equations are also important dynamical systems in understanding the

high-dimensional deterministic chaos in complex systems, such as Globally Coupled Maps and Coupled Map Lattices (GCML) [2]. Although discrete-time dynamical systems have been a mathematically informative platform for the analysis of non-linear integrable systems, such as discrete-time solitons [3], the methodology on the derivation of the difference equations from the original differential equations is considered decisive for the feasibility of the analysis; the transformation from continuous to discrete-time dynamical systems should preserve essential dynamics that constitute the complexity of the original model.

For example, the logistic equation and its discretized logistic map are usually described as follows, for the one-dimensional real variable x(t) with time t, environmental capacity K and strength of self-feedback r:

Logistic equation 
$$L(x(t)): \frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{\kappa}\right)$$
 (1)

Logistic map: 
$$x(t+1) = x(t) + rx(t)\left(1 - \frac{x(t)}{\kappa}\right)$$
 (2)

However, the logistic map (2) is known to exhibit complex bifurcation of periodicity and chaotic behavior along with the augmentation of the parameter r [4], which is not observed in the continuous-time model (1). Although the logistic map is useful for investigating coupled chaotic systems, such dynamics could be considered as mathematical by-products resulting from the naive discretization apart from the original ecological implication of the logistic equation. For equation (2) to rigorously reproduce the original solution known as the logistic/sigmoid curve, one needs to employ the following model proposed by Morishita [5]:

Morishita's logistic map: 
$$x(t + \Delta t) = \frac{(1+a)x(t)}{1+bx(t)}$$
, (3)

where  $\Delta t > 0$  is the time difference,  $a = e^{r\Delta t} - 1$  and b = a/K.

The model (2) and (3) could be integrated with a generalized discretization form of (1), using the time difference  $l = (m-1)\tau$  with a positive real number  $0 < \tau$  and natural number *m*, such that

$$x(t + \Delta t) = x(t) + r\Delta t \left( x(t) - \frac{1}{\kappa} x(t) x(t+l) \right) + O(\tau).$$
(4)

Since the sigmoid curve x(t) has the upper bound K, so is the variation in (4), *i.e.*, for a sufficiently large  $\tau$ ,  $\left\{x(t + \Delta t) - x(t) - r\Delta t\left(x(t) - \frac{1}{K}x(t)x(t+l)\right)\right\}/\tau < K$ holds. By taking the infinitesimal limit  $\tau \to +0$ , (4) converges to (1) when l = 0 and  $\Delta t \to +0$ ; to (2) when l = 0 and  $\Delta t = 1$ ; and to (3) when l = 1 and  $\Delta t \ll 1$ . (Note that the actual dynamics of (3) perfectly coincide with the trajectory of (1) with all ranges of  $\Delta t > 0$ , which is not fully expressed in (4) but these differences converge to the same ultra-discretized model (10).) This means that the choice of the discretization parameters such as l and  $\Delta t$  could influence the nature of discretized dynamics which may not preserve the original characteristics, especially in non-linear systems. Recently developing analysis on ultradiscretization deals with such challenges to extract essential systems that are discrete not only in time but also the values of variables that are representative of the original complexity and mathematical invariants (*e.g.*, [6][7]). Since the numerical simulation of non-linear dynamical systems with simple discretization such as the Euler method from (1) to (2) may profoundly alter the characteristics of the model, seeking universal mathematical structure among differential equations, difference equations and ultra-discretized digital equations is essential for the proper computation [8].

# 2 Ultradiscretization of logistic models

#### 2.1 Ultradiscretization of the logistic equation

We consider the ultradiscretization of the logistic equation starting from the rigorously discretized form (3) with  $\Delta t = 1$  and a < K. We focus on the sigmoidal growth of x(t) starting from x(0) > 0 and converges to K. Using an arbitrary parameter  $\varepsilon > 0$ , we consider the following transformation from x(t) to X(t):

$$x(t) = x(0)e^{\frac{X(t) - X(0)}{\varepsilon}}$$
(5)

$$1 + a = e^{\frac{R}{\varepsilon}}, R > 0 \tag{6}$$

$$b = e^{\frac{1}{\varepsilon}}, Q > 0 \tag{7}$$

We then obtain

$$X(t+1) = X(t) + R - \varepsilon \log\left(e^{\frac{0}{\varepsilon}} + e^{\frac{X(t) - Q - X(0) - \varepsilon \log x(0)}{\varepsilon}}\right),\tag{8}$$

which by taking the limit  $\varepsilon \to +0$ , converges to the ultra-discretized form

$$\lim_{\epsilon \to +0} X(t+1) = X(t) + R - \max[0, X(t) - Q - X(0)].$$
(9)

We choose the representative values A and B ( $0 \le A \le K$ ,  $0 \le B \le K$ , A + B = K) for the parameters R and Q that formally express the convergence to the environmental limit K as follows:

$$X(t+1) = X(t) + A - \max[0, X(t) - B].$$
(10)

From the relation between (3) and (4), this result coincides with the ultradiscretization of the limited case of (4) with the specification of parameters  $\Delta t = 1$ , l=1 and  $\tau \rightarrow +0$ . The relationships between equations (1), (3) and (10) are depicted in Fig. 1.

### 2.2 Ultradiscretization of the neuron model with a sigmoid function

We consider another discrete-time system based on a sigmoid function commonly used in neural network models and show that the ultra-discretized dynamics coincide with that of (10). Let us consider the sigmoid function  $S(x; r, \theta, K) \coloneqq K/(1 + e^{-r(x-\theta)})$  with the parameters  $r, \theta, K > 0$ , which is known as a basic neuron model with K=1 and also as the solution of (1) in the form of  $x(t) \coloneqq S(t) = K/(1 + e^{-r(t-\theta)})$ , where  $\theta \coloneqq \log(K/x(0) - 1)/r$ .

We consider a simple element of discrete-time system conventionally used in the self-recurrent neural network models with the time-dependent difference term  $\Delta x(t) > 0$  such as

$$x(t+1) = S(x(t))$$
$$= x(t) + \Delta x(t).$$
(11)

Using (3), it is known that

$$S(t + \Delta t) = \frac{(1+a)S(t)}{1+bS(t)},$$
(12)

and substituting t and  $\Delta t$  with x(t) and  $\Delta x(t)$ , respectively, we obtain

$$S(x(t) + \Delta x(t)) = S(x(t+1)) = \frac{(1+a')S(x(t))}{1+b'S(x(t))},$$
(13)

where  $a' = e^{r\Delta x(t)} - 1$  and b' = a'/K. Note that a' and b' are dependent on t and not on  $\Delta t$ . Through the transformation  $t + 1 \rightarrow t$ , we derive a simpler form

$$x(t+1) = \frac{(1+a')x(t)}{1+b'x(t)}.$$
(14)

We apply the transformation  $x(t) = e^{\frac{X(t)}{\varepsilon}}$  for the ultradiscretization, which transforms (14) into the following:

$$\varepsilon \log e^{X(t+1)/\varepsilon} = \varepsilon \log \left\{ (1+a')e^{X(t)/\varepsilon} \right\} - \varepsilon \log \left\{ 1+b'e^{X(t)/\varepsilon} \right\}.$$
(15)

Performing the parameters transformation  $a^{"} = \varepsilon' \log(1 + a')$  and  $b^{"} = -\varepsilon^{"} \log b'$ with  $\varepsilon' > 0$  and  $\varepsilon^{"} > 0$ , and taking the limit  $\varepsilon \to 0$  on both sides, we obtain

$$X(t+1) = X(t) + a^{"} - max[0, X(t) - b^{"}].$$
(16)

- (i) In the case of  $X(t) \le b^{"}$ , (16) becomes  $X(t+1) = X(t) + a^{"}$ . By definition,  $a^{"} = \varepsilon' r \Delta x(t)$ , therefore by taking  $\varepsilon' \propto \Delta x(t)^{-1}$ , (16) can be described as X(t+1) = X(t) + A with an arbitrary constant A > 0.
- (ii) In the case of  $X(t) \ge b^{"}$ , (16) becomes  $X(t+1) = a^{"} + b^{"} = \varepsilon^{"} \log\left(K\frac{1+a'}{a'}\right)$  by assuming  $\varepsilon' = \varepsilon^{"}$ . Then by taking  $\varepsilon^{"} = \frac{B}{\log(K/a')}$  with B = K A > 0, X(t+1) = K remains invariant for all t.

Based on the operations (i) and (ii), taking the limit  $\varepsilon', \varepsilon'' \to 0$  on (16) results in

Ultra-discretized sigmoidal map S'(X(t)):

$$X(t+1) = X(t) + A - max[0, X(t) - B],$$
(17)

which exactly coincides with (10). This means that the logistic equation (1) and the self-recurrent sigmoidal map (11) preserve the same mathematical structure in the ultradiscretized form. From this perspective, it may be more appropriate to use the combination of the sigmoidal map (11) than the logistic map (2) to numerically simulate the mathematical features of coupled logistic equations. The relationship between (11) and (17) is depicted in Fig. 1.



**Fig. 1.** (a), (b), (c): Relationships between the equations L(x(t)) (1), a sigmoid curve x(t)=S(t) and its ultradiscretization (10). (d), (e): relationships between the equations (11) and its ultradiscretization (17). (f): discrete-time development of the ultra-discretized models (10) and (17). Red arrows represent ultradiscretization, and blue arrows correspond to the same coordinate transformation. Example parameters were chosen as r = 1, x(0) = 1 for (a) and (b); r = 0.2, x(0) = 13.83 for (d); and K = 100, A = 15, X(0) = 1 for (c), (e) and (f).

### 2.3 Analogy between neural network and agroecological dynamics

Considering the self-recurrent coupled sigmoidal maps as an essential discrete model of the logistic equation (1) opens a way to reinterpret the neural networks from the perspective of the ecological model of interacting species. In this section, we explore ways to express autonomous ecological dynamics and human interventions for its management in reference to discrete-time neural network models such as [9], based on the general modalities used in a wide range of farming methods reviewed in [10].

Crop and weed interactions are important factors in the community dynamics of agroecological systems, which entail symbiotic and competitive relationships that promote and hinder the growth of each species and produce a variety of succession dynamics. The most intuitive way to express such complex interactions is to incorporate the connection functions  $V \coloneqq \left\{ v_{ij} \left( x_j(t); x_i(t) \right) \right\}_{i,j=1}^n$  between sigmoidal maps, such as

$$x_{i}(t+1) = S\left(\sum_{j=1}^{n} v_{ij}\left(x_{j}(t); x_{i}(t)\right); r_{i}, \theta_{i}, K_{i}\right),$$
(18)

where  $S(y; r_i, \theta_i, K_i) \coloneqq K_i/(1 + e^{-r_i(y-\theta_i)})$ , and  $v_{ij}(x_i(t), x_j(t)) > 0$  represents facilitation and  $v_{ij}(x_j(t); x_i(t)) < 0$  inhibition from  $x_j(t)$  to  $x_i(t+1)$  using  $x_i(t)$  as a parameter. In ecosystems,  $v_{ij}(:)$  are generally non-linear that are typical in allelopathic interactions between crops and weeds [11], but the simplest linearized form  $v_{ij}(x_j(t); x_i(t)) \approx w_{ij}x_j(t)$  converges to the connection matrix of a conventional neural network (*e.g.*, [9]), which provides

$$x_{i}(t+1) = S\left(\sum_{j=1}^{n} w_{ij} x_{j}(t); r_{i}, \theta_{i}, K_{i}\right).$$
(19)

Here, the variables  $\{x_i(t)\}_{i=1}^n$  could represent the growth rate of *n* different individual organisms, as well as the population size of *n* species, in the latter case  $w_{ij}$  can be considered as the mean-field approximation of the total interactions between the *j*-th and *i*-th species.

The connection matrix  $W \coloneqq \{w_{ij}\}_{i,j=1}^{n}$  can represent facilitative  $(w_{ij} > 0)$ and competitive  $(w_{ij} < 0)$  effects from  $x_j(t)$  to  $x_i(t + 1)$ , and the asymmetry in Wcan also reproduce periodic vegetation succession patterns in an analogous way to the dynamical associative memory in neural network (*e.g.*,[12]). The dynamical modification of W could also serve as an evolutionary model of the community structure.

There exist ecological interactions that affect the environmental capacity  $K_i$  more than the inter-species interactions  $w_{ij}$  in (19), which are classically known as the physiological optimum of a single species and the ecological optima of a community of multiple species [13]. Such effects could be incorporated as  $K_i(\bar{E})$  with the mean environmental condition  $\bar{E}$ , such as

$$K_i(\bar{E}) := K'_i N(\bar{E}; \mu_i, \sigma_i^{opt}), \tag{20}$$

where  $\overline{E}$  represents the mean environmental parameters that affect plant growth such as temperature, humidity, luminosity, etc., and  $N(\cdot; \mu_i, \sigma_i^2)$  is the probabilistic density function of a normal distribution with the mean  $\mu_i$  and standard deviation  $\sigma_i^{opt}$ . The superscript *opt* specifies the physiological (*phy*) and ecological (*eco*) optimizations, which is generally known as  $\sigma_i^{phy} < \sigma_i^{eco}$ , *i.e.*, the relative superiority of ecological optimum in a marginal environment [10]. For the practical fitting of the model, we can choose the constant  $K'_i$  according to the unit of measurement such as biomass quantity.

Ecosystems dynamics are dependent on the past states and often exhibit hysteresis such as regime shifts with inherent mechanisms [14]. A general form to incorporate time-delayed feedback in (19) can be expressed as the following:

$$x_{i}(t+1) = S\left(\sum_{j=1}^{n} w_{ij} H\left(\left\{x_{j}(t-d)\right\}_{d=0}^{t}\right); r_{i}, \theta_{i}, K_{i}\right).$$
(21)

The hysteresis function  $H(\cdot)$  can be approximately decomposed to each past time step using multiple linear regression such as

$$H\left(\left\{x_{j}(t-d)\right\}_{d=0}^{t}\right) \approx x_{j}(t) + \sum_{d=1}^{t} h'(d) x_{j}(t-d),$$
(22)

with a series of coefficients  $\{h'(d)\}_{d=1}^t$ . If we assume a gradual decrease of the past influence, such as  $h'(1) > h'(2) > \cdots > h'(t)$ , one of the simplest and plausible ways to interpret ecologically important situations (*e.g.*, allelopathic residual effects in the soil [15]) is to express them as the exponential decay such as  $h'(d) \coloneqq h^d$  with the *n*-th power of the attenuation coefficient  $0 \le h < 1$ , which provides

$$x_i(t+1) = S\left(\sum_{j=1}^n w_{ij} \sum_{d=0}^t h^d x_j(t-d); r_i, \theta_i, K_i\right).$$
(23)

Among the time-delayed feedbacks in (23), self-recurrent negative feedback is especially important to explain the hysteresis such as monocropping failure or replant difficulty (*e.g.*, [16][17]), which can be expressed as the refractory term of the *i*-th element that is introduced in a chaotic neuron model [9]:

$$x_i(t+1) = S\left(\sum_{j=1}^n w_{ij} \sum_{d=0}^t h_f^d x_j(t-d) - \alpha \sum_{d=0}^t h_r^d x_i(t-d); r_i, \theta_i, K_i\right), \quad (24)$$

where  $\alpha$  is the scaling coefficient of the refractoriness, and the attenuation coefficient h is distinguished between the feedback and refractory terms as  $h_f$  and  $h_r$ , respectively. The refractory term can also represent context-dependent human interventions such as thinning harvest, density-dependent weed & pest control (*e.g.*, [18]) and other management strategies that consider the growth history of the target  $x_i$ .

A more general and systematic form of human interventions for the management, as well as external periodic factors such as seasonal microclimate changes, can be additionally incorporated as the external inputs term to (24), such that

$$x_{i}(t+1) = S(\sum_{j=1}^{m} u_{ij} \sum_{d=0}^{t} h_{e}^{d} E_{j}(t-d) + \sum_{j=1}^{n} w_{ij} \sum_{d=0}^{t} h_{f}^{d} x_{j}(t-d) - \alpha \sum_{d=0}^{t} h_{r}^{d} x_{i}(t-d); r_{i}, \theta_{i}, K_{i}),$$
(25)

with the interaction matrix  $U \coloneqq \{u_{ij}\}$  between the *j*-th external input  $E_j$  (j = 1, ..., m) and the *i*-th element  $x_i$  and its attenuation coefficient  $h_e$ <sup>\*</sup>.

The model (25) is one of the simplest linear approximations of complex ecological interactions with the structure of coupled sigmoidal maps, which converges to a neural network with intermittent chaotic behavior [9][12]. We analogically call it the Chaotic Ecological Network (CEN) and analyze its dynamics from agroecological perspectives in the following section.

#### 2.4 Analysis of the dynamics in Chaotic Ecological Network

Let us simulate example dynamics of CEN (25) with the parameters n = 3,  $K_i = K = 100$ ,  $r_i = 0.2$ ,  $\theta_i = \log(K_i/x_i(0) - 1)/r_i = 22.9756$ ,  $h_e = h_f = 0.1$ , and the initial

<sup>\*</sup> For simplicity, U was defined as a unit matrix in the simulations of Fig. 4.

values with a relative saturation of  $x_1(t)$  such that  $(x_1(0), x_2(0), x_3(0)) = (K, 0, -K)$ . The values of  $E_j$ ,  $\alpha$  and  $h_r$  differ and are specified in each simulation. We define the interaction matrix  $W = \{w_{ij}\}_{i,j=1}^{n=3}$  as follows, to set the ecological niches of n=3 different species (or organisms) as three stable fix points represented with the patterns  $K \cdot (p_1^{k'}, p_2^{k'}, p_3^{k'})$ , where  $\{p_{i=1,2,3}^{k'=1}\} = (1,0,0), \{p_{i=1,2,3}^{k'=2}\} = (0,1,0)$  and  $\{p_{i=1,2,3}^{k'=3}\} = (0,0,1)$ , based on the definition of the Hebbian learning rule in neural networks [12]:

$$w_{ij} = \frac{1}{3} \sum_{k'=1}^{3} (2p_i^{k'} - 1) (2p_j^{k'} - 1).$$
<sup>(26)</sup>

Note that (26) only expresses competitive relationships among  $x_i(t)$ , because  $w_{ij} < 0$  if  $i \neq j$ . The phase diagrams with the analysis of periodicity, stability and converging niches of the dynamics are shown in Fig. 2, with varying ranges of the refractoriness  $\alpha$  and  $h_r$ .



**Fig. 2.** Periodicity (**Left**), stability (**Middle**) and converging niche patterns (**Right**) of CEN. Refractoriness parameter ranges  $h_r = [0, 0.5]$ ,  $\alpha = [0, 10000]$  were used for the simulations, without external inputs *i.e.*,  $E_j = 0$ . After cutting t=10000 initial steps, the dynamics of additional t=1000 steps were classified with the colors representing periodicity (**Left**, according to the color bar); positive and negative maximum Lyapunov exponents with red and blue, respectively (**Middle**, according to the calculation method in [12]); and the mean value patterns of  $x_i(t)$  (**Right**) that showed the dominance of  $x_1(t)$  with blue (the mean values of  $x_1(t) \ge K/2$ ,  $x_2(t)$  and  $x_3(t) < K/2$ ) and the inferior and competitive growth patterns with red (the mean values of  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t) < K/2$ ).

Based on the analysis in Fig. 2, it is possible to classify most of the simulated dynamics of CEN into four Areas on the  $h_r - \alpha$  plane:

- Area 1. Monoculture dominant condition, where a single crop  $x_1(t)$  continuously and stably grows more than the others: Periodicity = 1; maximum Lyapunov exponents < 0; mean  $x_1(t)$  is dominant ( $\ge K/2$ ) over  $x_2(t)$  and  $x_3(t)$  (< K/2).
- Area 2. Monoculture in competition, where a single crop  $x_1(t)$  continuously and stably grows but at an inferior level and in strong competition with the other species: Periodicity = 1; maximum Lyapunov exponents < 0; mean  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  remain inferior to K/2.

- Area 3. Rotational succession, where multiple species form stable limit cycles of varying periods: Periodicity > 1; maximum Lyapunov exponents < 0; mean  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  remain inferior to K/2.
- Area 4. Chaotic itinerancy, where multiple species follow unstable chaotic trajectories among different niche patterns: Periodicity > 1000; maximum Lyapunov exponents > 0; mean  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  remain inferior to K/2.

For simplicity, we considered  $x_1(t)$  as the growth of the target crop species,  $x_2(t)$  and  $x_3(t)$  as the competing crop and/or weed species. The dominance of the target crop in Area 1 and the inferior growth in the other Areas qualitatively reflect the magnitude relationship of single-crop productivity between physiological and ecological optima [10]. The complex behaviors of Areas 3 & 4 may correspond to the diversity of vegetation succession important in agroecology, which can be leveraged with human operations for the augmentation of biodiversity and ecosystem functions [19]. Typical dynamics of the four Areas are simulated in Fig. 3.



**Fig. 3.** Typical dynamics of Areas 1-4. Simulated parameters  $(h_r, \alpha)$  are: (0.1, 0) for Area 1, (0.48, 300) for Area 2, (0.155, 3400) for Area 3, and (0.15, 7000) for Area 4. The "perturbed" species take the initial values (K, 0, -K) + 1 that are slightly different (+1) from the other species. The example of Area 1 represents the successful dominance of a single crop, while that of 2 corresponds to the inferior growth of multiple species under strong competition. The example of Area 3 shows high periodicity (more than 1000) but stability against the perturbation, while that of 4 follows chaotic dynamics where initial perturbations are amplified to the system level.

Although the interaction matrix *W* is set to be only competitive in (26), the model can still introduce the effect of symbiosis with the use of (20). We can also incorporate symbiotic relationships by modifying the *W* itself, such as by defining a new interaction matrix  $W' = \{w'_{ij}\}_{i,j=1}^{n=3}$ :

$$w_{ij}' = \frac{1}{2} \sum_{k'=1}^{2} (2p_{i}'^{k'} - 1) (2p_{j}'^{k'} - 1), \qquad (27)$$

where  $\{p'_{i=1,2,3}^{k'=1}\} = (1,1,0)$  and  $\{p'_{i=1,2,3}^{k'=2}\} = (0,0,1)$ , which expresses the symbiosis between species 1 and 2, while species 3 remains competitive.

Furthermore, in analogy to the external inputs that could stabilize chaotic neural networks (*e.g.*, [20][21]), the effects of external inputs  $\{E_j\}$  with agroecological contexts such as harvesting and application of agrochemicals, as well as the time-delayed feedback effects of these interventions, can be investigated.

Examples of these extended features of community structure and responses to feedback inputs are simulated in Fig. 4. The overall results imply the utility of symbiotic interactions and context-dependent negative feedbacks for the amelioration of productivity and stabilization of periodicity in agroecological contexts.



**Fig. 4.** Typical dynamics with symbiotic interactions and external inputs. (a): Area 1 dynamics with  $(h_r, \alpha) = (0.1, 0)$ , with competitive interactions W(26) and its modified W'(27) with symbiotic interactions between the species 1 and 2. (b): Area 2 dynamics with  $(h_r, \alpha) = (0.48, 300)$  and W, using  $K_i = K = 100$  and  $K_i = K' = 10K$  for the representation of maximum growth rate of species with physiological and ecological optima, respectively, under a marginal environment in (20). (c): Area 1 dynamics with  $(h_r, \alpha) = (0.1, 0)$  and W', with the negative external inputs that represent the thinning harvest of the crop species 1: if  $x_1(t) = K$  then  $E_1(t + 1) = -4x_1(t)$ ; and suppression of the weed species 2: if  $x_2(t) \ge K/100$  then  $E_2(t + 1) = -1.75x_2(t)$ ; which produce periodic dynamics close to actual harvesting and weeding cycles. (d): Area 4 dynamics with  $(h_r, \alpha) = (0.15, 7000)$  with W', with time-delayed negative external inputs to the species  $1 \& 2: E_i(t + 1) = -500(x_i(t) - x_i(t - 1)), i = 1,2$ , which increases the stability of the chaotic orbits. The feedback parameters were chosen for the visibility of plots.

# **3** Elementary ultra-discrete automaton

Although it is technically difficult to ultra-discretize CEN (25), the complexity of CEN can be essentially reduced to the dynamics of the single chaotic neuron model [9], which may be possible to further explore essential underlying structure using the ultra-

discretized sigmoidal map (17). Here we consider the following ultra-discrete automaton that further incorporates self-recurrent refractoriness in S'(X(t)):

Ultra-discretized sigmoidal map with refractoriness S''(X(t)):

$$X(t+1) = \mu \left( X(t) + A - max \left[ 0, X(t) - B + max[0, X(t) - C] \right] \right),$$
(28)

where 0 < A, B < K, A + B = K,  $C \ge (\mu - 1)K$ . The natural number parameter  $\mu$  is inspired by the classical tent map that exhibits a fully developed chaotic regime in  $\mu =$ 2, and we later show that (28) produces orbital instability when  $\mu \ge 2$ . At the limit of  $A \rightarrow 0$  with the parameter range  $0 < \mu \le 2$ , (28) converges to the tent map on the interval [0, B+C] if  $B \ge C$ . In the case of B < C, a plateau  $X(t + 1) = \mu K$  exists in the return map (e.g., Fig. 5 (b)), which converges plural orbits of X(t) into a single value  $\mu K$ .

The incorporation of refractoriness in (28) is similar to the nested max()

operations proposed in the ultra-discretized model of cryptic oscillations in Lotka-Volterra equations with additional negative feedbacks using Holling's disc equation [22]. Indeed, the type III disc equation is qualitatively similar to the sigmoid function, and the type I disc equation exactly matches the ultra-discretized sigmoidal map (20) (see Fig. 6 Top).

The dynamics of (28) are analyzed in Fig. 5. Typical cases that exhibit unstable limit cycles were simulated with  $\mu = 1$  in (a) and  $\mu = 2$  in (b), with the periodicity analysis of  $\mu = 2$  in (c) and actual dynamics in (d). Since deterministic chaos on the set of real values is known to contain infinite numbers of unstable limit cycles, *i.e.*, the skeleton of chaos [23], the unstable limit cycles that remain in the ultra-discretized models could be considered as the essential mathematical structure of chaos that is preserved in the ultra-discretized limit. In (e), the local temporal stability of a trajectory was judged with an ultra-discrete version of the Lyapunov exponent, namely the one-sided local Lyapunov exponent (*LE*) defined as follows:

$$LE(X(t)) \coloneqq \log(\left|S''(X(t)) - S''(X(t) + 1)\right|).$$
<sup>(29)</sup>

This means that the one-sided digital perturbation (+1) to the variable X(t) is amplified in t + 1 if LE > 0, diminished to zero if  $LE = -\infty < 0$ , and remains invariant if LE = 0. Note that LE = 0 if  $\mu = 1$ , and  $LE \neq 0$  if  $\mu \ge 2$  in (28). Therefore,  $\mu \ge 2$  could be also considered as providing an ultra-discretized analog of the logistic map (2), since the smooth interpolation of (28) with a real-value resolution qualitatively converges the dynamics to that of (2). A multi-dimensional spectrum  $LE_i(\mathbf{X}(t))$  (1, ..., i, ..., n) can be defined on the vector variable  $\mathbf{X}(t) = (X_1(t), ..., X_n(t))$  and its ultra-discrete map  $\mathbf{X}(t + 1) = F(\mathbf{X}(t))$  using the *i*-th row of the *n*-dimensional identity matrix  $I_i$ , such that

$$LE_{i}(\boldsymbol{X}(t)) \coloneqq \log(|F(\boldsymbol{X}(t)) - F(\boldsymbol{X}(t) + I_{i})|).$$
(30)

Since ultra-discretized models are known to form a *max-plus* algebra that can be qualitatively considered as the transformation of additions (+) and multiplications

( $\times$ ) in the original model into the *max*( $\cdot$ ) and *plus* (+) operations, respectively, the coupled version of (28) could be proposed as follows:

$$F(X(t), X^{+}(t), X^{-}(t)) = max[X(t) + A, X^{+}(t) + A^{+}]$$

$$-\max\left[0,\max[X(t) - B, X^{-}(t) - B^{-}] + \max[0,X(t) - C]\right],$$
(31)

$$X(t+1) = \max[0, \mu F(X(t), X^{+}(t), X^{-}(t))], \qquad (32)$$

where  $X^+(t)$  and  $A^+ > 0$  represent a positively interacting term and  $X^-(t)$  and  $B^- > 0$  a negative one. An example of (32) producing intermittent dynamics between partially stable and unstable states is simulated in Fig. 6 Bottom, which is qualitatively similar to the chaotic itinerancy reported both in chaotic neural networks[9][12] and GCML[2][24], within the constraint of periodicity subject to the possible number of discrete values that X(t) can take on the interval [0, K + C].

The rationale behind the newly introduced models (28) and (32) can be found in relation to the expansion of Holling's disc equations. Through the inverse transformation of ultradiscretization, we can obtain the underlying discrete model of (32) as follows, with the parameters  $a, a^+ > 0$  and  $0 < b, b^-, c < 1$ :

$$x(t+1) = 1 + \frac{a(x(t))^{\mu} + a^{+}(x^{+}(t))^{\mu}}{1 + (b(x(t))^{\mu} + b^{-}(x^{-}(t))^{\mu})(1 + c(x(t))^{\mu})}.$$
(33)

This model reduces to a single uncoupled element that corresponds to the inverse ultradiscretization of (28), such as

$$x(t+1) = \frac{a(x(t))^{\mu}}{1+b(x(t))^{\mu}(1+c(x(t))^{\mu})}.$$
(34)

On the other hand, the three types of Holling's disc equations  $\mathcal{H}'(\cdot)$  and  $\mathcal{H}(\cdot)$  as a discrete-time system of the variable  $y(t) \ge 0$  can be formulated as follows (*e.g.*, [25]), using the parameters *a*, *b* and *K*:

Type I: 
$$y(t+1) = \mathcal{H}'(y(t)) := \min(K, ay(t))$$
 (35)  
Type II and III:

Type II and III:

$$y(t+1) = \mathcal{H}(y(t)) := \frac{a(y(t))^{\mu}}{1+b(y(t))^{\mu}},$$
  
where  $\mu = 1$  (type II),  $\mu > 1$  (type III). (36)

The model (34) can therefore be interpreted as the expansion of types II and III incorporating refractoriness at the saturation stage of y(t), which we call type II -R and III-R, respectively:

Type II -R and III-R:

$$y(t+1) = \mathcal{H}_{R}(y(t)) := \frac{a(y(t))^{\mu}}{1+b(y(t))^{\mu}(1+c(y(t))^{\mu})},$$
  
where  $\mu = 1$  (type II-R),  $\mu > 1$  (type III-R). (37)

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Additionally, we extract a typical non-linear response of single chaotic neuron based on (24) as follows, using y(t) as the internal state of sigmoidal output x(t):

$$y(t+1) = \eta(t+1) + \zeta(t+1), \tag{38}$$

$$\eta(t+1) = h_f \,\eta(t) + wx(t) + w'x'(t), \tag{39}$$

$$\zeta(t+1) = h_r \,\zeta(t) - \alpha x(t), \tag{40}$$

$$x(t+1) = S(y(t+1); r, \theta, K),$$
(41)

where w and w' are positive connection coefficients and x'(t) is the output from another neuron.

The relationships between type I, II, III, III-R, III-R and the chaotic neuron (38)-(41) are depicted in Fig. 6 Top, which shows qualitative correspondence between type III-R and the chaotic neuron.



**Fig. 5.** Dynamics of the ultra-discretized sigmoidal map with refractoriness S''(X(t)) (28). Interval maps between X(t) and X(t + 1) are depicted in (a): with parameters ( $\mu$ , A, B, C, K) =

(1, 20, 80, 0, 100) that can be considered as an ultra-discretized analog of the tent map, and (b): ( $\mu$ , *A*, *B*, *C*, *K*) = (2, 20, 80, 100, 100) as that of the logistic map. In (a) and (b), the initial values were taken for all X(t = 0) = [0, 200], which converge to unstable limit cycles (red spiderweb plot) and partially stable limit cycles (blue) with respect to the stability LE(X(t)). (c): Periodicity of the model (b) with respect to the initial values X(t = 0) = [0, 200]. (d): Temporal dynamics X(t) of the model (b) from t = 0 to 100 and the initial values X(t = 0) = [0, 200], with a color gradient from cyan (X(t) = 0) to magenta (X(t) = 200). (e) The same

dynamics as (d) with the color distinction between LE(X(t)) > 0 (red) and LE(X(t)) < 0(blue). The blue regions in (e) correspond to the plateau  $X(t + 1) = \mu K = 200$  in (b), except the right endpoint that gives LE(X(t) = 100) > 0.



**Fig. 6. Top:** Relationship between Holling's disc equations (35)(36) and their expanded models (37) (38) corresponding to S''(X(t)) (28), and chaotic neuron model (38)-(41). Parameters  $h_f = h_r = 0$ ,  $x'(t) = S(y'(t); r, \theta, K)$ , y'(t) = r'y(t), r' > 1 are used for the simulation of the chaotic neuron. Parameter values were chosen for visibility. **Bottom:** Example dynamics of coupled S''(X(t)) (32). Three automata (blue, red, and green) are negatively coupled with one-directional circular connections, *i.e.*  $A^+ = 0$  and  $B^- > 0$ , by which dynamics of blue, red, and green negatively affect red, green, and blue, respectively. Solid circles represent stable points concerning the three-dimensional version of *LE* (30). Common parameters are K = C = 1000, and A = (2, 7, 20), B = (998, 993, 980) and  $B^- = (980, 998, 993)$  are used for blue, red, and green automata, respectively.

### 4 Discussion

CEN (25) is presented as the reinterpretation of chaotic neural networks from agroecological perspectives, which essentially employs the same model for different phenomenological classifications of the dynamics such as Area 1-4 in Fig. 2. Only simple linearized interactions were simulated in this article, and it can incorporate other nonlinearity specific to ecosystems using the general form of interactions (18) and (21), such that

$$x_{i}(t+1) = S\left(\sum_{j=1}^{n} \nu_{ij}\left(H(\{x_{i}(t-d)\}_{d=0}^{t}); H\left(\{x_{j}(t-d)\}_{d=0}^{t}\right)\right); r_{i}, \theta_{i}, K_{i}\right).$$
(42)

In contrast to the differential equation models, the merit of using discrete and ultradiscrete models is the facility of calculation and practicality of incorporation of many realistic processes and interactions, such as the reproduction of individual organisms and the occurrence of human interventions, which are essentially discrete phenomena. Instead of modeling every detail with a real-value resolution, discrete models can reduce non-essential features for the effective abstraction with an appropriate resolution.

On the other hand, proper discretization of a continuous model cannot be *a priori* defined with a fixed formula and requires thorough mathematical comparison. For example, consider a typical Lotka-Volterra equation with a conserved quantity  $Q_c := cx + by - d \log x - a \log y$  as follows:

$$\frac{dx}{dt} = ax - bxy, \frac{dy}{dt} = cxy - dy.$$
(43)

Then the following coupled discrete-time maps could be derived using a variant of (4):

$$x(t+1) = \frac{(1+\Delta ta)x(t)}{1+\Delta tby(t)}, y(t+1) = \frac{(1+\Delta tcx(t))y(t)}{1+\Delta td}.$$
(44)

Through the appropriate parameters transformation and the limit operation analogous to (5)-(9), it is possible to derive the following ultra-discretized form:

$$X(t+1) = X(t) + A - \max[0, Y(t) - B],$$
  

$$Y(t+1) = \max[0, X(t) + C] + Y(t) - D.$$
(45)

However, the models (44) and (45) do not reproduce the closed limit cycle of the original differential equations (43), but rather express divergent oscillation (*i.e.*, expanding spirals on *x*-*y* and *X*-*Y* planes, results not shown) as the discretization time step  $\Delta t$  increases. It is therefore necessary to consider other forms of discretization to examine the property of the integrable system (43) based on the conserved quantity [26].

The difficulty of interpreting discrete models also exists in quantitative analysis, both in temporal scales and variable values. Parameter fitting and its optimization methods need to be explored according to the focus of the research. On the other hand, robust system-level features such as the typology of ecological regime shifts and scale-free phenomena (*e.g.*, [19][27]) could be more accessible with qualitative analysis, which may further contribute to the management model of essential ecosystem services and hierarchical modeling of the comprehensive biosphere.

The ultra-discrete automaton (32) was actually inspired by unsuccessful attempts to directly ultra-discretize CEN (25). It is proposed as a synthetic model to serve as a stepstone for further investigation, in the same line with the arbitrary modification of dLV to derive a simple ultra-discrete automaton with intermittent dynamics such as in [22]. Nevertheless, the proposed model captures essential characteristics of coupled chaotic systems concerning the intermittent fluctuation of orbital stability commonly reported in coupled logistic maps and chaotic neural networks. It should be noted that the original chaotic neuron model was not based on the rigorous discretization of differential equation neuron model (such as Hodgkin-Huxley and FitzHugh-Nagumo models) either, but rather based on a qualitative observation using *a priori* defined discrete-time formal neuron model [9]. Since the proposed ultra-discrete model (32) conserves essential characteristics of chaos that are generally not integrable, further analysis needs to extend the methodology beyond the conventional framework restricted to the conserved quantity of integrable non-linear systems.

The commonality of dynamics analyzed between neural networks and ecological models could potentially provide a starting point for the integration of knowledge between the vast non-linear classification capacity of deep learning (*e.g.*, [28]) and the complexity of ecosystem dynamics that could provide access to untapped utilities in agroecology.

## 5 Conclusion

We investigated the relationship between the logistic equation and the sigmoidal map with the light of ultradiscretization that conserves the essential dynamics including chaos. The results suggest that the conventional logistic map (2) may not be appropriate to interpret as a simple discretized form of the logistic equation (1), but rather as the one that belongs to a class of model which can be better presented with a self-recurrent sigmoidal function incorporating refractoriness. This insight will further bring clarification in ecological and complex systems modeling on the distinction between technical by-products and computational rationales for mathematically sound simulations. An ultra-discrete model based on the expansion of Holling's disc equations was proposed (32), which qualitatively captured the characteristics of intermittent dynamics known as chaotic itinerancy commonly reported in chaotic neural networks and GCML.

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#### References

- 1. Hugo, F.: From growth equations for a single species to Lotka-Volterra equations for two interacting species. In: Ecological Modelling and Ecophysics, pp. 1-41, IOP Publishing (2020).
- 2. Kaneko, K.: Overview of coupled map lattices. Chaos 2, 279 (1992).
- Hirota, R., Tsujimoto, S., Imai, T.: Difference Scheme of Soliton Equations. In: Christiansen, P.L., Eilbeck, J.C., Parmentier, R.D. (eds) Future Directions of Nonlinear Dynamics in Physical and Biological Systems. NATO ASI Series, vol. 312. Springer, Boston, MA (1993).
- 4. May, R.M.: Simple mathematical models with very complicated dynamics". Nature 261: 459-467 (1976).
- Morishita, M.: The fitting of the logistic equation to the rate of increase of population density. Res. Popul. Ecol. (1965) 52 (1965).
- Tokihiro, T.: Ultradiscrete Systems (Cellular Automata). In: Discrete Integrable Systems, Lect. Notes Phys. 644, 383–424, Springer-Verlag Berlin Heidelberg (2004).
- 7. Tokihiro, T., Takahashi, D., Matsukidaira, J. and Satsuma, J.: Phys. Rev. Lett., 76, 3247 (1996).
- Takahashi, D.: Continuous, Discrete, Ultradiscrete Waves. RIMS Kôkyûroku, vol. 1191, pp. 104-111 (2001).
- Aihara, K., Takabe, T., Toyoda, M.: Chaotic neural networks. Physics Letters A, vol. 144, Issues 6-7, pp. 333-340 (1990).
- Funabashi, M.: Synecological farming: Theoretical foundation on biodiversity responses of plant communities. Plant Biotechnology, 2016, vol. 33, issue 4, pp. 213-234 (2016).
- Belz, R.G.: Allelopathy in crop/weed interactions-an update. Pest Manag Sci. 63 (4) pp. 308-326 (2007).
- 12. Funabashi, M.: Synthetic Modeling of Autonomous Learning with a Chaotic Neural Network. International Journal of Bifurcation and Chaos, vol. 25, no. 04, 1550054 (2015).
- Putman, R.J., Wratter, S.D. Principle of Ecology. University of California Press, USA (1984).
- Beisner, B., Haydon, D. and Cuddington, K.: Alternative stable states in ecology. Frontiers in Ecology and the Environment, vol. 1, issue 7, pp.376-382 (2003).
- Xuan, T.D., Tawata, S., Khanh, T.D. and Chung, I.M.: Decomposition of Allelopathic Plants in Soil. Journal of Agronomy and Crop Science, 191, pp.162-171 (2005).
- Lü, L.H., Srivastava, A. K., Shen, Y.L., Wu, Q. S.: A Negative Feedback Regulation of Replanted Soil Microorganisms on Plant Growth and Soil Properties of Peach. Not Bot Horti Agrobo, 47 (1) pp. 255-261 (2019).
- Tang, B., Dong, Y., Wu, K., He, M., Liu, J., Yin, F., Zhang, W. and Gong, M.: Crop rotation alleviates replant failure in Panax notoginseng (Burkill) F.H. Chen by changing the composition but not the structure of the microbial community," Plant Soil Environ., vol. 66, pp. 493-9 (2020).
- Shyu, E., Pardini, E.A., Knight, T.M. and Caswell, H.: A seasonal, density-dependent model for the management of an invasive weed. Ecological Applications, 23, pp.1893-1905 (2013).
- Funabashi, M.: Augmentation of Plant Genetic Diversity in Synecoculture: Theory and Practice in Temperate and Tropical Zones. In: Nandwani, D. (eds) Genetic Diversity in Horticultural Plants. Sustainable Development and Biodiversity, vol. 22, Springer, Cham. (2019).
- Shibasaki, M. and Adachi, M.: Response to external input of chaotic neural networks based on Newman-Watts model. In: The 2012 International Joint Conference on Neural Networks (IJCNN), Brisbane, QLD, Australia, pp. 1-7 (2012).

- Yamamoto, S., Ushio, T.: Robust Stabilization of Chaos via Delayed Feedback Control. In: Hashimoto, K., Oishi, Y., Yamamoto, Y. (eds) Control and Modeling of Complex Systems. Trends in Mathematics. Birkhäuser, Boston, MA (2003).
- Willox, R., Ramani, A., Grammaticos, B.: A discrete-time model for cryptic oscillations in predator-prey systems. Physica D: Nonlinear Phenomena, vol. 238, issue 22, pp. 2238-2245 (2009).
- Cvitanović, P.: Periodic orbits as the skeleton of classical and quantum chaos. Physica D: Nonlinear Phenomena, vol. 51, issues 1–3, pp. 138-151 (1991).
- 24. Balmforth, N. J., Jacobson, A. and Provenzale, A.: Synchronized family dynamics in globally coupled maps. Chaos 9, 738 (1999).
- Dawes, J.H.P., Souza, M.O.: A derivation of Holling's type I, II and III functional responses in predator–prey systems. Journal of Theoretical Biology, vol. 327, pp. 11-22 (2013).
- Hirota, R., Iwao, M., Ramani, A., Takahashi, D., Grammaticos, B., Ohta, Y.: From integrability to chaos in a Lotka-Volterra cellular automaton. Physics Letters A, vol. 236, issues 1–2, pp. 39-44 (1997).
- 27. Funabashi, M.: Human augmentation of ecosystems: objectives for food production and science by 2045. npj Sci Food 2, 16 (2018).
- Amari, S.: Any Target Function Exists in a Neighborhood of Any Sufficiently Wide Random Network: A Geometrical Perspective. Neural Computation, vol. 32, no. 8, pp. 1431-1447 (2020).

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